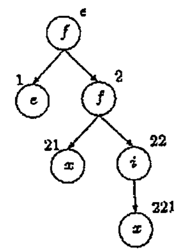

Fig. 3.1. Tree representation of $t = f(e, f(x, i(x)))$.

Using a standard numbering of the nodes of the tree by strings of positive integers (as illustrated in the example), we can refer to positions in a term. In our example, position ϵ (the empty string) refers to the symbol f on the top level, and position 2 refers to the symbol f that occurs as second argument of the top-level f. The subterm of t at position 2 is $f(x, i(x))$, and the subterm of t at position 22 is $i(x)$. More formally, notions like position and subterm can be defined by induction on the structure of terms.

Definition 3.1.3 Let Σ be a signature, X be a set of variables disjoint from Σ, and $s, t \in T(\Sigma, X)$.

1. The set of **positions** of the term s is a set $\mathcal{P}os(s)$ of strings over the alphabet of positive integers, which is inductively defined as follows:
   - If $s = x \in X$, then $\mathcal{P}os(s) := \{\epsilon\}$, where ϵ denotes the empty string.
   - If $s = f(s_1, \ldots, s_n)$, then
   $$\mathcal{P}os(s) := \{\epsilon\} \cup \bigcup_{i=1}^{n} \{ip \mid p \in \mathcal{P}os(s_i)\}.$$
   The position ϵ is called the **root position** of the term s, and the function or variable symbol at this position is called the **root symbol** of s. The **prefix order** defined as
   $$p \leq q \ \text{ iff } \ \text{there exists } p' \text{ such that } pp' = q$$
   is a partial order on positions. We say that the positions p, q are **parallel** ($p \parallel q$) iff p and q are incomparable with respect to $\leq$. The position p is **above** q if $p \leq q$ and p is **strictly above** q if $p < q$ (**below** is defined analogously).
2. The **size** $|s|$ of a term s is the cardinality of $\mathcal{P}os(s)$.

3. For $p \in \mathcal{P}os(s)$, the **subterm of s at position p**, denoted by $s|_p$, is defined by induction on the length of p:
   $$\begin{aligned} s|_\epsilon &:= s, \\ f(s_1, \ldots, s_n)|_{iq} &:= s_i|_q. \end{aligned}$$
   Note that, for $p = iq$, $p \in \mathcal{P}os(s)$ implies that s is of the form $s = f(s_1, \ldots, s_n)$ with $i \leq n$.
4. For $p \in \mathcal{P}os(s)$, we denote by $s[t]_p$ the term that is obtained from s by **replacing the subterm at position p by t**, i.e.
   $$\begin{aligned} s[t]_\epsilon &:= t, \\ f(s_1, \ldots, s_n)[t]_{iq} &:= f(s_1, \ldots, s_i[t]_q, \ldots, s_n). \end{aligned}$$
5. By $\mathcal{V}ar(s)$ we denote the set of **variables occurring** in s, i.e.
   $$\mathcal{V}ar(s) := \{x \in X \mid \text{there exists } p \in \mathcal{P}os(s) \text{ such that } s|_p = x\}.$$
   We call $p \in \mathcal{P}os(t)$ a **variable position** if $t|_p$ is a variable.

For the term t of the above example, $\mathcal{P}os(t) = \{\epsilon, 1, 2, 21, 22, 221\}$, $t|_{22} = i(x)$, $t[e]_2 = f(e, e)$, $\mathcal{V}ar(t) = \{x\}$, and $|t| = 6$. Note that the size of t is just the number of nodes in the tree representation of t. The set of positions of a term is obviously closed under taking prefixes, i.e. if $q \in \mathcal{P}os(t)$ then $p \in \mathcal{P}os(t)$ for all $p \leq q$. The following lemma states some useful rules for computing with positions and subterms.

Lemma 3.1.4 *Let s, t, r be terms and p, q be strings over the positive integers.*

1. *If $pq \in \mathcal{P}os(s)$, then $s|_{pq} = (s|_p)|_q$.*
2. *If $p \in \mathcal{P}os(s)$ and $q \in \mathcal{P}os(t)$, then*
   $$\begin{aligned} (s[t]_p)|_{pq} &= t|_q, \\ (s[t]_p)[r]_{pq} &= s[t[r]_q]_p. \end{aligned}$$
3. *If $pq \in \mathcal{P}os(s)$, then*
   $$\begin{aligned} (s[t]_{pq})|_p &= (s|_p)[t]_q, \\ (s[t]_{pq})[r]_p &= s[r]_p. \end{aligned}$$
4. *If p and q are parallel positions in s (i.e. $p \parallel q$), then*
   $$\begin{aligned} (s[t]_p)|_q &= s|_q, \\ (s[t]_p)[r]_q &= (s[r]_q)[t]_p. \end{aligned}$$